

PENALTY APPROXIMATION FOR NON SMOOTH CONSTRAINTS IN VIBROIMPACT

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ABSTRACT. We examine the penalty approximation of the free motion of a material point in an angular domain; we choose an over-damped penalty approximation, and we prove that if the first impact point is not at the vertex, then, the limit of the approximation exists and is described by Moreau's rule for anelastic impacts.

The proofs rely on validated asymptotics and use some classical tools of the theory of dynamical systems.

1. INTRODUCTION

Mathematical results relative to the convergence of a penalty approximation of impact problems have been obtained by several authors when the energy is conserved; see for instance [13], [14], and also [3], [1] [2], [12], [6], [8], [7], [5], [4] and [10].

When energy may be lost at impact, the convergence of the penalty approximation has been treated in [11] in the case of a convex set of constraints with smooth boundary. In this article, we defined a penalty approximation for which the limit solution satisfies a Newton condition at impact: the normal component of the velocity is reversed and multiplied by a restitution coefficient $e \in]0, 1]$ and the tangential component is transmitted.

So far, we are not aware of any mathematical results on the convergence of the penalty approximation when the boundary is not smooth and energy can be lost at impact.

Here, we study the penalty approximation of the motion of a free particle constrained to stay inside an angular domain of \mathbb{R}^2 : we choose a class of penalty approximations for which the restitution coefficient vanishes in the limiting problem and we characterize precisely the limit of the sequence of solutions of the approximated problem when the first impact does not take place at the corner.

We compare our results to the ones given by the selection rule of Moreau [9], and we find complete agreement.

2. THE FIRST PART OF THE MOTION AND THE MATHEMATICAL STRATEGY

Let us describe more precisely the problem and the method of solution.

Given $\bar{\theta} \in (0, \pi)$, we let K be the set

$$(2.1) \quad K = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \leq 0 \text{ and } x_1 \cos \bar{\theta} + x_2 \sin \bar{\theta} \leq 0\}.$$

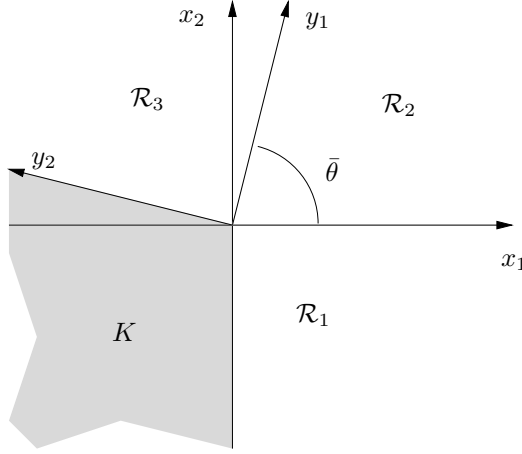


FIGURE 1. The regions and the coordinates in the plane.

The closure of the complement of K is partitioned into three regions:

$$\begin{aligned}\mathcal{R}_1 &= \{(x_1, x_2) : x_1 \geq 0 \text{ and } x_2 \leq 0\}, \\ \mathcal{R}_2 &= \{(x_1, x_2) : x_2 \geq 0 \text{ and } -x_1 \sin \bar{\theta} + x_2 \cos \bar{\theta} \leq 0\} \\ \mathcal{R}_3 &= \{(x_1, x_2) : x_1 \cos \bar{\theta} + x_2 \sin \bar{\theta} \geq 0 \\ &\quad \text{and } -x_1 \sin \bar{\theta} + x_2 \cos \bar{\theta} \geq 0\}.\end{aligned}$$

In each of these regions, the projection onto K , which is known to be a contraction, takes different forms:

$$P_K x = \begin{cases} (0, x_2)^T, & \text{if } x \in \mathcal{R}_1, \\ 0, & \text{if } x \in \mathcal{R}_2, \\ (-x_1 \sin \bar{\theta} + x_2 \cos \bar{\theta})(-\sin \bar{\theta}, \cos \bar{\theta})^T, & \text{if } x \in \mathcal{R}_3. \end{cases}$$

The penalty approximation used in [11] is defined as follows: we define a function G of two arguments $u \in \mathbb{R}^2$ and $v \in \mathbb{R}^2$ by

$$G(u, v) = \begin{cases} \frac{(v \cdot (u - P_K u))(u - P_K u)}{|u - P_K u|^2}, & \text{if } u \notin K, \\ 0, & \text{otherwise.} \end{cases}$$

Then, the penalized approximation to the impact problem in K , in the absence of exterior forces is given by

$$(2.2) \quad \ddot{u}_k + 2\alpha\sqrt{k}G(u_k, \dot{u}_k) + k(u_k - P_K u_k) = 0.$$

In this formulation, the number k is the stiffness of the spring which describes the reaction of the wall, and the choice of the scale \sqrt{k} is the exact choice which ensures convergence in the smooth case, as k tends to infinity. Here, we choose $\alpha > 1$: it is the over-damped choice and it will lead to a vanishing restitution coefficient as we shall see.

Take initial conditions given by

$$(2.3) \quad \begin{aligned} u_k(0) &= (0, x_2(0))^T, \quad x_2(0) < 0, \\ \dot{u}_k(0) &= (\dot{x}_1(0), \dot{x}_2(0))^T, \quad \dot{x}_1(0) > 0, \dot{x}_2(0) > 0. \end{aligned}$$

These initial conditions mean that at time $t = 0$, the representative point of the system is on the boundary of K , in region \mathcal{R}_1 , and outgoing as well as taking the direction of the corner. In particular, this choice of initial conditions means that the first impact time is $t = 0$.

The roots of the characteristic equation

$$\xi^2 + 2\alpha\xi + 1 = 0$$

of the over-damped equation

$$(2.4) \quad \ddot{y} + 2\alpha\dot{y} + y = 0$$

are given by the formulas

$$(2.5) \quad \Delta = \alpha^2 - 1, \quad \xi_1 = -\alpha + \sqrt{\Delta}, \quad \xi_2 = -\alpha - \sqrt{\Delta}.$$

Both ξ_1 and ξ_2 are negative.

As long as the representative point of the system lies in \mathcal{R}_1 , we perform the change of variables $r(t) = x_1(t) \geq 0$ and $s(t) = x_2(t) \leq 0$. In these new coordinates, (2.2) becomes the decoupled system

$$\begin{aligned} \ddot{r} + 2\alpha\sqrt{k}\dot{r} + kr &= 0, \\ \ddot{s} &= 0. \end{aligned}$$

Its solution is given explicitly by

$$(2.6) \quad r(t) = \frac{\dot{r}(0)}{2\sqrt{\Delta k}} \left(e^{\xi_1 t \sqrt{k}} - e^{\xi_2 t \sqrt{k}} \right),$$

$$(2.7) \quad s(t) = s(0) + t\dot{s}(0).$$

For all positive t , $r(t)$ given by (2.6) remains strictly positive; s reaches the value 0 at the time

$$(2.8) \quad t_0 = -s(0)/\dot{s}(0).$$

Therefore, at the boundary between regions \mathcal{R}_1 and \mathcal{R}_2 we have

$$(2.9) \quad \begin{aligned} s(t_0 - 0) &= 0, \quad \dot{s}(t_0 - 0) = \dot{s}(0), \\ r(t_0 - 0) &= \frac{\dot{r}(0)}{2\sqrt{\Delta k}} \left(e^{\xi_1 t_0 \sqrt{k}} - e^{\xi_2 t_0 \sqrt{k}} \right), \\ \dot{r}(t_0 - 0) &= \frac{\dot{r}(0)}{2\sqrt{\Delta}} \left(\xi_1 e^{\xi_1 t_0 \sqrt{k}} - \xi_2 e^{\xi_2 t_0 \sqrt{k}} \right). \end{aligned}$$

In order to study the motion in region \mathcal{R}_2 , we use polar coordinates i.e. $u_k = re^{i\theta}$ and we define scaled functions and variables R , Θ and τ by

$$r(t) = \eta R(\tau)/\sqrt{k}, \quad \tau = (t - t_0)\sqrt{k}, \quad \eta = e^{\xi_1 t_0 \sqrt{k}/2}, \quad \Theta(\tau) = \theta(t).$$

We have represented in Fig. 2 the numerically computed trajectories (dotted or dashed lines) and the vector field of the ordinary differential equation for R and \dot{R} .

In the new variables, the system under consideration becomes

$$(2.10) \quad \ddot{R} - \frac{E(1 - \varepsilon)^2}{R^3} + 2\alpha\dot{R} + R = 0.$$

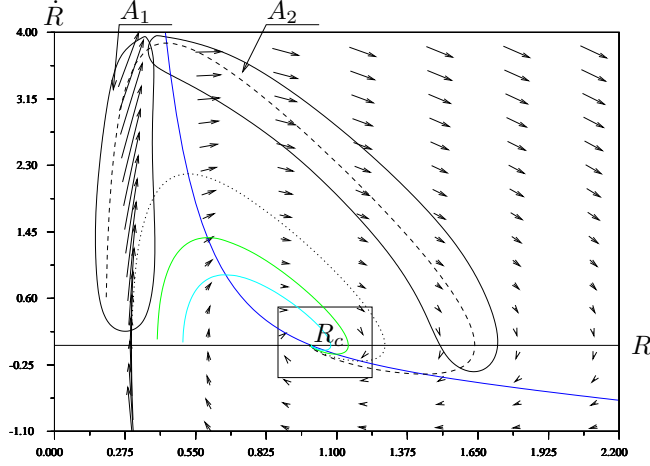


FIGURE 2. The phase portrait for the R equation, and several trajectories of solutions in the R, \dot{R} plane. A_1 : region of the first asymptotic (section 3); A_2 : region of the second asymptotic (section 5).

with

$$(2.11) \quad \dot{\Theta} = \frac{\sqrt{E}(1-\varepsilon)}{R^2},$$

and the detailed derivation of these equations is performed in subsection 3.1. In equations (2.10) and (2.11), $\varepsilon = o(1)$ and E is a fixed number depending only on the initial conditions and α . The representation given in Fig.2 will help us to explain how the solution of (2.10) behaves, with appropriate consequences on the angle Θ .

In region A_1 , R decreases somewhat and then increases, \dot{R} increases from a size equivalent to $C\eta$ to a size equivalent to C/η in that same region. The dominant terms in equation (2.10) are \ddot{R} and $E(1-\varepsilon)^2/R^3$; therefore, we are led to the problem

$$(2.12) \quad \ddot{R}_1 - \frac{E}{R_1^3} = 0, \quad R_1(0) = R(0), \quad \dot{R}_1(0) = \dot{R}(0).$$

We study the solution R_1 of (2.12) in subsection 3.2, as well as the evolution of the function Θ_1 satisfying

$$\dot{\Theta}_1 = \sqrt{E}/R_1^2;$$

this can be done explicitly, thanks to the simple structure of (2.12). In subsection 3.3, we study the kernel of the linearized (2.12) at R_1 , as a preparation for the validation of this first asymptotic, a task which is completed in 3.4 on the interval $[0, \tau_1]$, where τ_1 is equal to η^{γ_1} , with γ_1 belonging to $(1, 2)$. We conclude section 3 by Proposition 3.8 which shows that R is equivalent to R_1 over $[0, \tau_1]$ and \dot{R} is equivalent to \dot{R}_1 over $[\eta^3, \tau_1]$. The proof is basically a consequence of the fixed point theorem with a number of technical estimates.

In Section 4, assuming $\bar{\theta} < \pi/2$, we are able to exploit the above equivalents and to prove that Θ , solution of (2.11), crosses through $\bar{\theta}$ at some time $\bar{\tau} < O(\eta^2)$. Moreover, our estimates enable us to describe the limit u_∞ of u_k as k tends to infinity. Let Π_1 be the orthogonal projection on $\{x_1 = 0\}$, and let Π_2 be the orthogonal projection on $\{x_1 \cos \bar{\theta} + x_2 \sin \bar{\theta} = 0\}$; then

$$u_\infty(t) = \begin{cases} u(0) + t\Pi_1\dot{u}(0) & \text{if } 0 \leq t \leq t_0, \\ (t - t_0)\Pi_2\Pi_1\dot{u}(t_0) & \text{if } t_0 \leq t. \end{cases}$$

If $\bar{\theta} \geq \pi/2$, the representative point of the system enters region A_2 of Fig. 2. We have to produce an asymptotic for the solution of (2.10); in this region, it is the linear part of this ordinary differential equation which is dominant; more precisely, let R_2 be the solution of

$$\ddot{R}_2 + 2\alpha\dot{R}_2 + R_2 = 0,$$

with R_2 and \dot{R}_2 respectively coinciding with R and \dot{R} at time $\tau_1 = \eta^{\gamma_1}$, where, now the interval of γ_1 is reduced to $(1, 4/3)$.

The validation of this ansatz is another consequence of the fixed point theorem for strict contraction, together with a number of technical estimates.

Finally, we use classical methods for dynamical systems and prove that the representative point of the system tends to $(R_c, 0)$ as time tends to infinity: R_c is a number which depends only on the initial conditions, α and ε . We combine the use of a Lyapunov functional and some elementary properties of the system to conclude that R remains bounded from above and away from 0 for all time after leaving A_2 . Observe that the Lyapunov functional gives scant information in regions A_2 and A_1 : there it takes values of order $1/\eta$.

With some technicalities in the case $\bar{\theta} = \pi/2$, it is possible to conclude that $\Theta(\tau)$ crosses $\bar{\theta}$ at some time $\bar{\tau}$ and to obtain precise equivalents for R , \dot{R} and $\dot{\Theta}$ at time $\bar{\tau}$. After this time, the representative point of the system (2.2) enters region \mathcal{R}_3 , and we conclude by Theorem 7.2 that the limit u_∞ of u_k is given by

$$u_\infty(t) = \begin{cases} u(0) + t\Pi_1\dot{u}(0) & \text{if } 0 \leq t \leq t_0, \\ 0 & \text{if } t_0 \leq t. \end{cases}$$

Moreau's rule is described as follows: at impact, the outgoing velocity is projected onto the tangent cone to the convex of constraints, and the motion proceeds with this new velocity. Thus, it can be seen that the over-damped penalty approximation agrees completely with Moreau's rule if the first impact does not take place at the corner, or very close to it, i.e. at a distance $O(1/\sqrt{k})$ from it.

We conjecture that the behavior described here still holds if there is a right hand side, and the convex is replaced by a set with convex corners, and smooth and not necessarily convex curves between corners. We also conjecture that the behavior of the limit of the over-damped penalized solution is the same in higher spatial dimension.

3. EQUATIONS OF THE MOTION AROUND THE CORNER: THE EARLIEST ASYMPTOTIC

3.1. Derivation of the scaled equation in \mathcal{R}_2 . After time t_0 , we arrive into region \mathcal{R}_2 , in which it is convenient to identify \mathbb{R}^2 and \mathbb{C} and to use polar coordinates,

i.e.

$$(3.1) \quad u_k = re^{i\theta}.$$

By continuity, the limits of $r(t)$ from the right and from the left as t tends to t_0 are identical; therefore:

$$(3.2) \quad r(t_0 + 0) = \frac{\dot{r}(0)}{2\sqrt{\Delta k}} \left(e^{\xi_1 t_0 \sqrt{k}} - e^{\xi_2 t_0 \sqrt{k}} \right).$$

At $t = t_0$,

$$(3.3) \quad \theta(t_0 + 0) = 0.$$

We differentiate once (3.1) with respect to time:

$$\dot{u}_k = \dot{r}e^{i\theta} + i\dot{\theta}re^{i\theta},$$

hence

$$(3.4) \quad \dot{r}(t_0 + 0) = \frac{\dot{r}(0)}{2\sqrt{\Delta}} \left(\xi_1 e^{\xi_1 t_0 \sqrt{k}} - \xi_2 e^{\xi_2 t_0 \sqrt{k}} \right),$$

$$(3.5) \quad \dot{\theta}(t_0 + 0) = \frac{\dot{\theta}(t_0 - 0)}{r(t_0 - 0)} = \frac{\dot{s}(0)2\sqrt{\Delta k}}{\dot{r}(0) \left(e^{\xi_1 t_0 \sqrt{k}} - e^{\xi_2 t_0 \sqrt{k}} \right)}.$$

Let us derive the differential equations satisfied by r and θ . In region \mathcal{R}_2 , the definition of the projection P_K implies that (2.2) can be written as

$$(3.6) \quad r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0,$$

$$(3.7) \quad \ddot{r} - r\dot{\theta}^2 + 2\alpha\sqrt{k}\dot{r} + kr = 0.$$

The motion has central acceleration, therefore it has a first integral: there exists a constant Γ such that

$$(3.8) \quad (r^2\dot{\theta})(t) = (r^2\dot{\theta})(t_0 + 0) = \Gamma,$$

and, according to (3.4) and (3.5), the value of Γ is given by

$$(3.9) \quad \Gamma = r(t_0 + 0)^2 \frac{\dot{s}(t_0 - 0)}{r(t_0 + 0)} = \frac{\dot{r}(0)\dot{s}(0)}{2\sqrt{\Delta k}} \left(e^{\xi_1 t_0 \sqrt{k}} - e^{\xi_2 t_0 \sqrt{k}} \right).$$

We substitute $\dot{\theta} = \Gamma/r^2$ into (3.7), and we find the equation in r

$$(3.10) \quad \ddot{r} - \frac{\Gamma^2}{r^3} + 2\alpha\sqrt{k}\dot{r} + kr = 0.$$

Let us find now appropriate changes of scale which will help us understand the behavior of r while u_k remains in region \mathcal{R}_2 . An important scale is defined by the number

$$(3.11) \quad \eta = e^{\xi_1 t_0 \sqrt{k}/2},$$

which is very small since ξ_1 is strictly negative. We perform the following changes of variables:

$$(3.12) \quad \tau = (t - t_0)\sqrt{k}, \quad r(t) = \eta R(\tau)/\sqrt{k}.$$

In these new variables, we have

$$\dot{r}(t) = \eta \dot{R}(\tau), \quad \ddot{r}(t) = \eta \sqrt{k} \ddot{R}(\tau),$$

so that equation (3.10) becomes, after a division by $\eta\sqrt{k}$

$$(3.13) \quad \ddot{R} - \frac{k\Gamma^2}{\eta^4 R^3} + 2\alpha\dot{R} + R = 0.$$

Let us define

$$(3.14) \quad E = \frac{\dot{r}(0)^2 \dot{s}(0)^2}{4\Delta},$$

$$(3.15) \quad \varepsilon = e^{(\xi_2 - \xi_1)t_0\sqrt{k}}.$$

With these notations, we find that Γ , given by (3.9), is equal to

$$(3.16) \quad \Gamma = \frac{\eta^2(1-\varepsilon)\sqrt{E}}{\sqrt{k}},$$

so that (3.13) can be rewritten as

$$(3.17) \quad \ddot{R} - \frac{E(1-\varepsilon)^2}{R^3} + 2\alpha\dot{R} + R = 0.$$

The initial values for (3.17) are given by

$$(3.18) \quad R(0) = \frac{\eta\dot{r}(0)(1-\varepsilon)}{2\sqrt{\Delta}},$$

and

$$(3.19) \quad \dot{R}(0) = \frac{\eta\dot{r}(0)\xi_1(1-\varepsilon\xi_2/\xi_1)}{2\sqrt{\Delta}}.$$

3.2. Ansatz for the motion in \mathcal{R}_2 . We shall use now an ansatz, namely, we state that the essential part in the left hand side of (3.17) is $\ddot{R} - E/R^3$. This comes from the fact that at time t_0 , E/R^3 is very large with respect to R and \dot{R} as can be checked from (3.18) and (3.19). Therefore, we first solve explicitly the equation

$$(3.20) \quad \ddot{R}_1 - \frac{E}{R_1^3} = 0, \quad R_1(0) = R(0), \quad \dot{R}_1(0) = \dot{R}(0).$$

If we multiply (3.20) by \dot{R}_1 and integrate, we find that

$$(3.21) \quad \dot{R}_1^2 + \frac{E}{R_1^2} = \dot{R}(0)^2 + \frac{E}{R(0)^2}.$$

Let us denote by

$$(3.22) \quad W = \dot{R}(0)^2 + \frac{E}{R(0)^2}$$

the value which appears on the right hand side of (3.21). Thanks to the change of variable $\rho = R_1^2$, equation (3.21) becomes:

$$(3.23) \quad \frac{W\dot{\rho}}{2\sqrt{W\rho - E}} = \pm W.$$

At the initial time, $\dot{\rho}(0)$ is strictly negative, so that in (3.23) we choose the minus sign on the right hand side, we integrate until a time τ_0 such that $\dot{\rho}$ vanishes, and we find

$$(3.24) \quad \rho(\tau) = \frac{E + (W\tau + \dot{R}(0)R(0))^2}{W}, \quad 0 \leq \tau \leq \tau_0.$$

The value of τ_0 is given by

$$(3.25) \quad \tau_0 = -\dot{R}(0)R(0)/W.$$

After τ_0 , we choose the plus sign in (3.23), and we find that

$$(3.26) \quad \rho(\tau)W - E = (\tau - \tau_0)^2 W^2.$$

Substituting the value of τ_0 into (3.26), we find that the general expression of the solution of (3.23) is given by

$$(3.27) \quad \rho(\tau) = \frac{E}{W} + W(\tau - \tau_0)^2.$$

The angle θ is defined by (3.1); we let

$$(3.28) \quad \Theta(\tau) = \theta(t).$$

We are only interested for the present moment in the principal part of Θ ; it is a function Θ_1 which satisfies the ordinary differential equation:

$$(3.29) \quad \dot{\Theta}_1(\tau) = \frac{\sqrt{E}}{\rho},$$

with the initial condition

$$(3.30) \quad \Theta_1(0) = 0.$$

We substitute the value of ρ given by (3.27) into (3.29) and we find that

$$(3.31) \quad \dot{\Theta}_1(\tau) = \frac{W\sqrt{E}}{E + (W\tau + \dot{R}(0)R(0))^2} = \frac{\sqrt{E}}{R_1(\tau)^2}$$

which we integrate immediately into

$$\Theta_1(\tau) = \arctan \frac{W\tau + \dot{R}(0)R(0)}{\sqrt{E}} - \arctan \frac{\dot{R}(0)R(0)}{\sqrt{E}}.$$

A more convenient way to write Θ_1 is the following:

$$(3.32) \quad \Theta_1(\tau) = \arctan \frac{W(\tau - \tau_0)}{\sqrt{E}} + \arctan \frac{W\tau_0}{\sqrt{E}}.$$

Thanks to (3.18) and (3.19), we can see that

$$(3.33) \quad W \sim \dot{s}(0)^2 \eta^{-2}.$$

But (3.33) and (3.25) imply that

$$(3.34) \quad \tau_0 = O(\eta^4).$$

This shows that we shall have to consider different cases: $\bar{\theta} < \pi/2$ and $\bar{\theta} \geq \pi/2$. If $\bar{\theta} < \pi/2$, we may suspect that u_k will exit region \mathcal{R}_2 at time approximately $t_0 + (\sqrt{E} \tan \bar{\theta})/(W\sqrt{k})$; while if $\bar{\theta} \geq \pi/2$, it is obvious that the ansatz is not sufficient: what will happen is that E/R^3 is no more large with respect to R and \dot{R} .

3.3. Kernel for the linearized equation. In order to go further, we have to validate asymptotics: consider therefore the linear differential equation obtained from the linearization of (3.20):

$$(3.35) \quad \ddot{z} + \frac{3Ez}{R_1^4} = 0.$$

We can obtain two linearly independent solutions by the following argument: if we differentiate the ordinary differential equation (3.20) with respect to time, we find that $z_1 = \dot{R}_1$ is a solution of (3.35); we seek another solution of (3.35) under the form

$$(3.36) \quad z_2 = z_1 T.$$

indeed, formula (3.27) gives the following form of R_1 :

$$(3.37) \quad R_1(\tau) = \sqrt{\frac{E}{W} + W(\tau - \tau_0)^2}.$$

We differentiate this relation with respect to τ , and we find that

$$(3.38) \quad \dot{R}_1(\tau) = \frac{W(\tau - \tau_0)}{R_1}.$$

The equation satisfied by T is

$$2\dot{z}_1\dot{T} + z_1\ddot{T} = 0.$$

We multiply this equation by z_1 , we integrate, and we find that, up to an irrelevant multiplicative constant, T satisfies the ordinary differential equation:

$$\dot{T} = \frac{1}{\dot{R}_1^2}.$$

This can be integrated exactly and we obtain

$$T = -\frac{E}{W^3(\tau - \tau_0)} + \frac{\tau}{W}.$$

According to definition (3.36), we have to multiply the above expression by \dot{R}_1 , for which we take expression (3.38). We obtain eventually

$$z_2 = -\frac{E}{W^2 R_1} + \frac{\tau(\tau - \tau_0)}{R_1}.$$

The Wronskian of z_1 and z_2 is readily calculated and is equal to

$$z_1\dot{z}_2 - z_2\dot{z}_1 = z_1^2\dot{T} = 1;$$

therefore z_1 and z_2 are independent. From here, we seek a kernel $K(\tau, \sigma)$ for $\tau \geq \sigma$ which satisfies the following conditions:

$$\begin{cases} \frac{\partial^2 K(\tau, \sigma)}{\partial \tau^2} + \frac{3EK(\tau, \sigma)}{R_1^4(\tau)} = 0, \\ K(\sigma, \sigma) = 0, \\ \frac{\partial K(\tau, \sigma)}{\partial \tau} \Big|_{\tau=\sigma} = 1, \end{cases}$$

under the form

$$K(\tau, \sigma) = a_1(\sigma)S_1(\tau) + a_2(\sigma)S_2(\tau).$$

Thanks to (3.3) and the definition of z_1 , we can see now that

$$K(\tau, \sigma) = -z_2(\sigma)z_1(\tau) + z_1(\sigma)z_2(\tau),$$

which can be rewritten as

$$(3.39) \quad K(\tau, \sigma) = \frac{\tau - \sigma}{R_1(\tau)R_1(\sigma)} \left[\frac{E}{W} + W(\sigma - \tau_0)(\tau - \tau_0) \right], \quad \tau \geq \sigma \geq 0.$$

We extend $K(\tau, \sigma)$ by 0 for $\sigma > \tau$. It is convenient to define

$$(3.40) \quad J(\tau, \sigma) = \frac{E}{W} + W(\sigma - \tau_0)(\tau - \tau_0);$$

with this notation, (3.39) becomes

$$(3.41) \quad K(\tau, \sigma) = \frac{(\tau - \sigma)J(\tau, \sigma)}{R_1(\tau)R_1(\sigma)},$$

and we may also remark that

$$(3.42) \quad R_1(\tau) = \sqrt{J(\tau, \tau)}.$$

Remark 3.1. The function $K(\tau, \sigma) \geq 0$ if $\tau \ll 1$: indeed the only possibility for K to be strictly negative is when the product $(\sigma - \tau_0)(\tau - \tau_0)$ is strictly negative, and $\sigma < \tau$; therefore, σ is smaller than τ_0 and τ is larger than τ_0 . Relations (3.25) and (3.33) imply that $W^2(\tau_0 - \sigma)(\tau - \tau_0)$ is estimated by $C\tau$. Therefore, if $\tau \ll 1$, $K(\tau, \sigma)$ is nonnegative.

3.4. Validation of the earliest asymptotic. With the help of the kernel K , we consider now the following problem: to find an interval $[0, \tau_1]$ and a mapping S_1 from this interval to \mathbb{R} such that $R_1 + S_1$ solves (3.17), with the initial conditions (3.18) and (3.19). As $R_1(0) = R(0)$ and $\dot{R}_1(0) = \dot{R}(0)$, $S_1(0)$ and $\dot{S}_1(0)$ have to vanish; therefore (3.17) can be rewritten as the following integral equation

$$(3.43) \quad S_1(\tau) = \mathcal{L}_1(S_1)(\tau) + G_1(\tau),$$

where G_1 is the function

$$(3.44) \quad G_1(\tau) = - \int_0^\tau K(\tau, \sigma)(2\alpha\dot{R}_1 + R_1)(\sigma) d\sigma,$$

and \mathcal{L}_1 is an integral operator defined by

$$(3.45) \quad \begin{aligned} \mathcal{L}_1(S_1)(\tau) = \int_0^\tau \left\{ K(\tau, \sigma) \left[\frac{3ES_1}{R_1^4} - \frac{E}{R_1^3} + \frac{E(1-\varepsilon)^2}{(R_1 + S_1)^3} - S_1 \right](\sigma) \right. \\ \left. + 2\alpha \frac{\partial K(\tau, \sigma)}{\partial \sigma} S_1(\sigma) \right\} d\sigma. \end{aligned}$$

Define

$$(3.46) \quad \tau_1 = \eta^{\gamma_1}, \quad 1 < \gamma_1 < 2.$$

Our purpose now is to prove that (3.43) has a unique solution on $[0, \tau_1]$ thanks to the strict contraction principle.

We equip the space of continuous functions on $[0, \tau_1]$ with the norm

$$(3.47) \quad \|S_1\| = \max\{|S_1(\sigma)|/R_1(\sigma) : 0 \leq \sigma \leq \tau_1\}.$$

The choice of the weight R_1 in the norm is natural since we expect R_1 to be the principal part of the solution; thus we expect that the relative error $(R - R_1)/R_1$ will be small: our norm measures precisely this relative error.

In order to apply the strict contraction principle, we estimate certain functions through a sequence of technical calculations.

Lemma 3.2. *For all large enough k , the expression*

$$(3.48) \quad I(\tau) = \frac{1}{R_1(\tau)} \int_0^\tau \frac{K(\tau, \sigma)}{R_1^3(\sigma)} d\sigma.$$

is bounded on $[0, 1]$. The bound will be called henceforth δ .

Remark 3.3. The expression $I(\tau)$ controls the nonlinear term in the integral equation (3.43), whose detail is given in (3.45).

Proof. We use the explicit expression of K to perform an estimate of I :

$$I(\tau) = \frac{1}{R_1^2(\tau)} \int_0^\tau \frac{\tau - \sigma}{R_1^4(\sigma)} \left[\frac{E}{W} + W(\sigma - \tau_0)(\tau - \tau_0) \right] d\sigma.$$

We introduce the notation

$$(3.49) \quad \kappa = \sqrt{E}/W,$$

and the change of variable

$$(3.50) \quad \tau = \tau_0 + \kappa y, \quad \sigma = \tau_0 + \kappa x.$$

The integral $I(\tau)$ is now given by

$$\begin{aligned} I(\tau) &= \frac{1}{J(\tau, \tau)} \int_0^\tau \frac{(\tau - \sigma)J(\tau, \sigma)}{J(\sigma, \sigma)^2} d\sigma \\ &= \frac{1}{E} \frac{1}{1 + y^2} \int_{-\tau_0/\kappa}^y \frac{(y - x)(1 + xy)}{(1 + x^2)^2} dx. \end{aligned}$$

We use the obvious inequalities

$$(3.51) \quad |1 + xy| \leq \sqrt{1 + x^2} \sqrt{1 + y^2},$$

$$(3.52) \quad |y - x| \leq |x| + |y|,$$

to infer that

$$|I(\tau)| \leq \frac{1}{E} \frac{|y|}{\sqrt{1 + y^2}} \int_{-\tau_0/\kappa}^y \frac{dx}{(1 + x^2)^{3/2}} + \frac{1}{E} \frac{1}{\sqrt{1 + y^2}} \int_{-\tau_0/\kappa}^y \frac{|x| dx}{(1 + x^2)^{3/2}}.$$

It is now clear that $|I(\tau)|$ is bounded independently of y , i.e. of τ by a certain number δ . \square

We shall prove now that for large enough k , we can apply the strict contraction principle to the mapping $S_1 \mapsto \mathcal{L}_1(S_1) + G_1$ on the interval $[0, \tau_1]$, defined by (3.46); for this purpose, we use the norm defined at (3.47), and we show the following result

Theorem 3.4. *For all $\gamma_1 \in (1, 2)$, and for all small enough $p \in (0, 1)$ there exists $k_0 > 0$ such that for all $k \geq k_0$, the mapping $S_1 \mapsto \mathcal{L}_1(S_1) + G_1$ leaves invariant the ball of center 0 and radius p and is a strict contraction in that ball. In particular, if R is the solution of (3.17), (3.18) and (3.19), we have the estimate*

$$(3.53) \quad \max_{0 \leq \tau \leq \eta^{\gamma_1}} \left| \frac{R(\tau) - R_1(\tau)}{R_1(\tau)} \right| \leq p.$$

The proof of this result depends on several estimates given in successive lemmas.

Lemma 3.5. *For all $\gamma_1 \in (1, 2)$, there exists a constant C such that*

$$(3.54) \quad \|G_1\| \leq C(\eta^{4\gamma_1-4} + \eta^{\gamma_1}).$$

Proof. By an integration by parts,

$$G_1 = 2\alpha K(\tau, 0)R_1(0) + \int_0^\tau \left(2\alpha \frac{\partial K}{\partial \sigma}(\tau, \sigma) - K(\tau, \sigma) \right) R_1(\sigma) d\sigma.$$

We estimate $|G_1(\tau)|/R_1(\tau)$: we first observe that

$$\frac{K(\tau, 0)R_1(0)}{R_1(\tau)} = \frac{\tau}{R_1^2(\tau)} \left[\frac{E}{W} - W\tau_0(\tau - \tau_0) \right];$$

We estimate $R_1^2(\tau)$ from below by E/W ; we also observe that $W^2\tau_0$ is bounded, thanks to estimates (3.25) and (3.33); therefore we have the following estimate, where we have used again remark 3.1:

$$(3.55) \quad \frac{K(\tau, 0)R_1(0)}{R_1(\tau)} \leq \tau(1 + W^2\tau_0\tau/E) = O(\eta^{\gamma_1}).$$

Next step is to calculate $\partial K(\tau, \sigma)/\partial \sigma$: we use formulas (3.39), (3.40) and (3.41) and we find that

$$\frac{\partial K}{\partial \sigma}(\tau, \sigma) = L_1(\tau, \sigma) + L_2(\tau, \sigma) + L_3(\tau, \sigma),$$

where the L_j 's are respectively given by

$$\begin{aligned} L_1(\tau, \sigma) &= -\frac{J(\tau, \sigma)}{R_1(\tau)R_1(\sigma)}, \\ L_2(\tau, \sigma) &= -\frac{(\tau - \sigma)(\sigma - \tau_0)J(\tau, \sigma)W}{R_1(\tau)R_1^3(\sigma)}, \\ L_3(\tau, \sigma) &= \frac{(\tau - \sigma)(\tau - \tau_0)W}{R_1(\tau)R_1(\sigma)}. \end{aligned}$$

Our aim is now to estimate the expressions

$$I_j(\tau) = \frac{1}{R_1(\tau)} \int_0^\tau |L_j(\tau, \sigma)| R_1(\sigma) d\sigma.$$

The first expression $I_1(\tau)$ is rewritten with the help of the change of variable (3.50) and becomes

$$I_1(\tau) = \frac{\kappa}{1 + y^2} \int_{-\tau_0/\kappa}^y |1 + xy| dx,$$

which we estimate as follows:

$$I_1(\tau) \leq \frac{\kappa}{\sqrt{1 + y^2}} \int_{-\tau_0/\kappa}^y \sqrt{1 + x^2} dx.$$

Since $\tau_0/\kappa \leq 1$ for k large enough, we can see that

$$\begin{aligned} \frac{1}{\sqrt{1 + y^2}} \int_{-\tau_0/\kappa}^y \sqrt{1 + x^2} dx &\leq \sqrt{2} \int_{-\tau_0/\kappa}^{\min(y, 0)} dx + \int_{\min(y, 0)}^y \frac{\sqrt{1 + x^2}}{\sqrt{1 + y^2}} dx \\ &\leq \sqrt{2} \left(\frac{\tau_0}{\kappa} + y \right). \end{aligned}$$

Therefore

$$I_1(\tau) \leq \sqrt{2} \tau.$$

With the change of variable (3.50), we may write the expression $I_2(\tau)$ as

$$I_2(\tau) = \frac{\kappa}{(1+y^2)} \int_{-\tau_0/\kappa}^y \frac{(y-x)|x||1+xy| dx}{1+x^2}.$$

We use (3.51) again together with

$$\frac{|x|}{\sqrt{1+x^2}} \leq 1,$$

and we find

$$I_2(\tau) \leq \frac{\kappa}{\sqrt{1+y^2}} \int_{-\tau_0/\kappa}^y (y-x) dx \leq \frac{W\tau^2}{2\sqrt{E} \sqrt{1+(\tau-\tau_0)^2/\kappa^2}}.$$

But for k large enough and $\tau \geq 0$,

$$\sqrt{1+(\tau-\tau_0)^2/\kappa^2} \geq \frac{\tau}{2\kappa};$$

indeed, if $\tau \geq 2\tau_0$, $\tau - \tau_0 \geq \tau/2$, and the inequality is clear; on the other hand, if $\tau \leq 2\tau_0$, for k large enough $\tau_0 \leq \kappa$, and the inequality also follows. Therefore, there exists a number C such that for all large enough k and all τ in $[0, \eta^{\gamma_1}]$ the following inequality holds:

$$I_2(\tau) \leq C\eta^{\gamma_1}.$$

The third expression is handled as follows:

$$I_3(\tau) = \frac{1}{R_1(\tau)} \int_0^\tau \frac{(\tau-\sigma)|\tau-\tau_0|W d\sigma}{R_1(\tau)} = \frac{|\tau-\tau_0|\tau^2 W}{2J(\tau, \tau)}.$$

If $0 \leq \tau \leq \tau_0 + \kappa$, we use the inequality $J(\tau, \tau) \geq E/W$ and we obtain

$$I_3(\tau) \leq \frac{\kappa^2 \tau^2 W^2}{2E} = O(\eta^2),$$

since $|\tau - \tau_0| \leq \max(\kappa, \tau_0) = \kappa$ for k large enough. On the other hand, for $\tau \geq \tau_0 + \kappa$ and for k large enough

$$\frac{\tau}{\tau - \tau_0} \leq 2,$$

and therefore, using the inequality $J(\tau, \tau) \geq W|\tau - \tau_0|^2$, we obtain

$$I_3(\tau) \leq \frac{|\tau - \tau_0|\tau^2 W}{2|\tau - \tau_0|^2 W} \leq \tau.$$

Thus, we have shown that

$$(3.56) \quad \frac{1}{R_1(\tau)} \int_0^\tau \left| \frac{\partial K}{\partial \sigma}(\tau, \sigma) \right| R_1(\sigma) d\sigma = O(\eta^{\gamma_1}).$$

There remains to estimate

$$(3.57) \quad \frac{1}{R_1(\tau)} \int_0^\tau K(\tau, \sigma) R_1(\sigma) d\sigma.$$

We rewrite (3.57) as

$$\frac{1}{R_1^2(\tau)} \int_0^\tau R_1^4(\sigma) \frac{\tau - \sigma}{R_1^4(\sigma)} \left[\frac{E}{W} + W(\sigma - \tau_0)(\tau - \tau_0) \right] d\sigma$$

and we find that thanks to remark 3.1

$$(3.58) \quad \begin{aligned} & \frac{1}{R_1(\tau)} \int_0^\tau |K(\tau, \sigma)| R_1(\sigma) d\sigma \\ &= \frac{1}{R_1(\tau)} \int_0^\tau K(\tau, \sigma) R_1(\sigma) d\sigma \leq \delta \max(R_1^4(0), R_1^4(\tau)) = O(\eta^{4\gamma_1-4}). \end{aligned}$$

Summarizing (3.58) with (3.55) and (3.56), we find estimate (3.54). \square

Next lemma enables us to estimate $\|\mathcal{L}_1(S_1)\|$ when $\|S_1\| \leq p < 1$.

Lemma 3.6. *Assume $\|S_1\| \leq p < 1$. For all $\gamma_1 \in (1, 2)$, there exists k_0 and C such that for all $k \geq k_0$ the following estimate holds:*

$$\|\mathcal{L}_1(S_1)\| \leq \frac{6E\delta p^2}{(1-p)^5} + \frac{2\varepsilon E\delta}{(1-p)^3} + pC(\eta^{4\gamma_1-4} + \eta^{\gamma_1})$$

Proof. The easiest part is the estimate on

$$\int_0^\tau \left(-K(\tau, \sigma) S_1(\sigma) + 2\alpha \frac{\partial K}{\partial \sigma}(\tau, \sigma) S_1(\sigma) \right) d\sigma.$$

We can see that the absolute value of this expression is estimated by

$$\|S_1\| \int_0^\tau \left(2\alpha \left| \frac{\partial K}{\partial \sigma}(\tau, \sigma) \right| + K(\tau, \sigma) \right) R_1(\sigma) d\sigma.$$

We recognize expressions which have already been estimated in (3.56) and (3.58). Therefore, it is immediate that

$$(3.59) \quad \begin{aligned} & \left| \frac{1}{R_1(\tau)} \int_0^\tau \left(-K(\tau, \sigma) S_1(\sigma) + 2\alpha \frac{\partial K}{\partial \sigma}(\tau, \sigma) S_1(\sigma) \right) d\sigma \right| \\ & \leq C\|S_1\|(\eta^{4\gamma_1-4} + \eta^{\gamma_1}). \end{aligned}$$

Next comes the slightly more complicated expression

$$(3.60) \quad \frac{1}{R_1(\tau)} \int_0^\tau K(\tau, \sigma) \frac{(2\varepsilon - \varepsilon^2)E}{(R_1 + S_1)^3(\sigma)} d\sigma.$$

For all $k > 0$, $2\varepsilon - \varepsilon^2$ is at most equal to 2ε ; therefore, if $\|S_1\| \leq p < 1$, then the absolute value of (3.60) is estimated by

$$\frac{2\varepsilon E}{R_1(\tau)} \int_0^\tau \frac{K(\tau, \sigma)}{R_1^3(\sigma)(1-p)^3} d\sigma.$$

Therefore, we can see that

$$(3.61) \quad \left| \frac{1}{R_1(\tau)} \int_0^\tau K(\tau, \sigma) \frac{(2\varepsilon - \varepsilon^2)E}{(R_1 + S_1)^3(\sigma)} d\sigma \right| \leq \frac{2\varepsilon E\delta}{(1-p)^3}.$$

The last and most complicated term contains the expression

$$\left(\frac{E}{(R_1 + S_1)^3} - \frac{E}{R_1^3} + \frac{3ES_1}{R_1^4} \right) (\sigma),$$

which can be rewritten thanks to Taylor's formula with integral remainder as

$$12E \int_0^1 \frac{S_1^2(\sigma)}{(R_1(\sigma) + sS_1(\sigma))^5} (1-s) ds.$$

The absolute value of this expression is estimated by

$$\frac{6Ep^2}{(1-p)^5 R_1(\sigma)^3}.$$

Therefore, we find that

$$(3.62) \quad \left| \frac{1}{R_1(\tau)} \int_0^\tau K(\tau, \sigma) \left(\frac{E}{(R_1 + S_1)^3} - \frac{E}{R_1^3} + \frac{3ES_1}{R_1^4} \right) (\sigma) d\sigma \right| \leq \frac{6E\delta p^2}{(1-p)^5}.$$

□

End of the proof of Theorem 3.4. The ball of radius p about 0 will be invariant by the mapping $S_1 \mapsto \mathcal{L}_1(S_1) + G_1$ provided that

$$(3.63) \quad \begin{aligned} & \frac{6E\delta p^2}{(1-p)^5} + \frac{2\varepsilon E\delta}{(1-p)^3} + pC(\eta^{4\gamma_1-4} + \eta^{\gamma_1}) \\ & + C(\eta^{4\gamma_1-4} + \eta^{\gamma_1}) \leq p. \end{aligned}$$

Choose p small enough for the following inequality to hold

$$\frac{6E\delta p}{(1-p)^5} \leq \frac{1}{2};$$

Choose then k so large that (3.63) holds.

Let us prove now that for an adequate choice of p and k_0 and for all $k \geq k_0$, the mapping \mathcal{L}_1 is a strict contraction: the easiest part of the estimate pertains to

$$\frac{1}{R_1(\tau)} \int_0^\tau \left(2\alpha(S_1 - \hat{S}_1)(\sigma) \frac{\partial K}{\partial \sigma}(\tau, \sigma) - (S_1 - \hat{S}_1)(\sigma) K(\tau, \sigma) \right) d\sigma$$

and it is clear from the proof of estimate (3.59) that

$$(3.64) \quad \begin{aligned} & \frac{1}{R_1(\tau)} \left| \int_0^\tau \left(2\alpha(S_1 - \hat{S}_1)(\sigma) \frac{\partial K}{\partial \sigma}(\tau, \sigma) - (S_1 - \hat{S}_1)(\sigma) K(\tau, \sigma) \right) d\sigma \right| \\ & \leq C \|S_1 - \hat{S}_1\| (\eta^{4\gamma_1-4} + \eta^{\gamma_1}). \end{aligned}$$

The second easiest term involves the difference

$$\frac{2\varepsilon - \varepsilon^2}{(R_1 + S_1)^3} - \frac{2\varepsilon - \varepsilon^2}{(R_1 + \hat{S}_1)^3}$$

which we estimate thanks to Taylor's formula:

$$6\varepsilon \int_0^1 \frac{|(\hat{S}_1 - S_1)(\sigma)|}{(R_1(\sigma) + S_1(\sigma) + s(\hat{S}_1(\sigma) - S_1(\sigma)))^4} ds \leq \frac{6\varepsilon \|\hat{S}_1 - S_1\|}{(1-p)^4 R_1(\sigma)^3}.$$

Therefore, the corresponding term in $\mathcal{L}_1(S_1) - \mathcal{L}_1(\hat{S}_1)$ contributes an estimate

$$(3.65) \quad \frac{6\varepsilon E\delta \|\hat{S}_1 - S_1\|}{(1-p)^4}.$$

The last and most complicated term involves the expression

$$\frac{3\hat{S}_1}{R_1^4} + \frac{1}{(R_1 + \hat{S}_1)^3} - \frac{3S_1}{R_1^4} - \frac{1}{(R_1 + S_1)^3}.$$

We use a Taylor expansion twice to rewrite this expression as

$$12(\hat{S}_1 - S_1) \int_0^1 \int_0^1 \frac{(S_1 + s(\hat{S}_1 - S_1)) ds' ds}{(R_1 + s'(S_1 + s(\hat{S}_1 - S_1)))^5}.$$

Therefore, the corresponding term is estimated by

$$\frac{12p\|\hat{S}_1 - S_1\|}{(1-p)^5 R_1(\sigma)^3}.$$

Thus, the norm of the corresponding contribution in $\mathcal{L}_1(\hat{S}_1) - \mathcal{L}_1(S_1)$ is estimated by

$$(3.66) \quad \frac{12pE\delta\|\hat{S}_1 - S_1\|}{(1-p)^5}.$$

Therefore, if we summarize the estimates (3.64), (3.65) and (3.66), we find that on a ball of radius $p < 1$ about 0, the Lipschitz constant of \mathcal{L}_1 is estimated by

$$(3.67) \quad C(\eta^{4\gamma_1-4} + \eta^{\gamma_1}) + \frac{6\varepsilon E\delta}{(1-p)^4} + \frac{12pE\delta}{(1-p)^5}.$$

If we choose p small enough for $12pE\delta/(1-p)^5$ to be less than or equal to $1/2$, it is clear that we can choose k_0 large enough for the sum of the remaining terms in (3.67) to be less than or equal to $1/4$.

Together with the conditions found above for the invariance of the ball of radius p about 0, we have shown the first part of theorem 3.4. We also infer from this proof that

$$(3.68) \quad \|S_1\| \leq C(\eta^{4\gamma_1-4} + \eta^{\gamma_1} + \varepsilon).$$

Its last assertion is an immediate consequence of the equivalence of (3.43) with (3.17), the definition of the norm, and the fact that the initial data coincide. \square

Remark 3.7. Let us observe that in the end of the proof of Theorem 3.4, we can take p arbitrarily small provided that k_0 is chosen large enough.

We conclude this section by the

Proposition 3.8. *Let γ_1 belong to $(1, 2)$, and let $\tau_1 = \eta^{\gamma_1}$. Then the following equivalences hold:*

$$(3.69) \quad R(\tau) \sim R_1(\tau) \text{ over } [0, \tau_1],$$

$$(3.70) \quad \dot{R}(\tau) \sim \dot{R}_1(\tau) \text{ over } [\eta^3, \tau_1].$$

Proof. The first statement is an almost immediate consequence of (3.68): we have

$$\frac{R(\tau)}{R_1(\tau)} = 1 + \frac{S_1(\tau)}{R_1(\tau)},$$

so that on $[0, \tau_1]$

$$\left| \frac{R(\tau)}{R_1(\tau)} - 1 \right| \leq \|S_1\| \leq C(\eta^{4\gamma_1-4} + \eta^{\gamma_1} + \varepsilon),$$

and (3.69) follows.

In order to compare \dot{R} and \dot{R}_1 , we write the differential equations that they satisfy:

$$\begin{aligned}\ddot{R} + 2\alpha\dot{R} &= -R + \frac{E(1-\varepsilon)^2}{R^3}, \\ \ddot{R}_1 + 2\alpha\dot{R}_1 &= \frac{E}{R_1^3} + 2\alpha\dot{R}_1.\end{aligned}$$

Therefore, if we subtract the second of these equations from the first, we deduce that

$$\ddot{R} - \ddot{R}_1 + 2\alpha(\dot{R} - \dot{R}_1) = -R + \frac{E(1-\varepsilon)^2}{R^3} - \frac{E}{R_1^3} - 2\alpha\dot{R}_1.$$

The initial data vanish.

We integrate because we want to estimate $\dot{R} - \dot{R}_1$:

$$\begin{aligned}(3.71) \quad & \dot{R}(\tau) - \dot{R}_1(\tau) \\ &= \int_0^\tau \exp(-2\alpha(\tau - \sigma)) \left\{ -R + \frac{E(1-\varepsilon)^2}{R^3} - \frac{E}{R_1^3} - 2\alpha\dot{R}_1 \right\}(\sigma) d\sigma.\end{aligned}$$

The next step is to estimate the integral on the right hand side of (3.71). We decompose this integral into three terms:

$$\begin{aligned}I_1 &= -2\alpha \int_0^\tau \exp(-2\alpha(\tau - \sigma)) \dot{R}_1(\sigma) d\sigma, \\ I_2 &= - \int_0^\tau \exp(-2\alpha(\tau - \sigma)) R(\sigma) d\sigma, \\ I_3 &= E \int_0^\tau \exp(-2\alpha(\tau - \sigma)) \left(\frac{(1-\varepsilon)^2}{R^3} - \frac{1}{R_1^3} \right) (\sigma) d\sigma.\end{aligned}$$

The first two integrals are very easy to estimate: for I_1 an integration by parts gives

$$I_1 = -2\alpha R_1(\tau) + 2\alpha e^{-2\alpha\tau} R_1(0) - 4\alpha^2 \int_0^\tau \exp(-2\alpha(\tau - \sigma)) R_1(\sigma) d\sigma.$$

Thanks to (3.37), on $[0, \tau_1]$, $R_1(\tau) = O(\eta^{\gamma_1-1})$. Therefore

$$(3.72) \quad |I_1| = O(\eta^{\gamma_1-1}).$$

For I_2 , the situation is even simpler since it can be readily seen that

$$(3.73) \quad |I_2| \leq (1+p) \int_0^\tau R_1(\sigma) \exp(-2\alpha(\tau - \sigma)) d\sigma = O(\eta^{2\gamma_1-1}),$$

where $p = \|S_1\|$.

There remains to estimate I_3 ; a straightforward calculation shows that

$$\left| \frac{(1-\varepsilon)^2}{R^3} - \frac{1}{R_1^3} \right| \leq \frac{2\varepsilon + p(3+3p+p^2)}{(1-p)^3} \frac{1}{R_1^3},$$

here $p = \|S_1\|$ is estimated at (3.68), so that

$$(3.74) \quad \left| \frac{(1-\varepsilon)^2}{R^3} - \frac{1}{R_1^3} \right| = O(\varepsilon + \eta^{\gamma_1} + \eta^{4(\gamma_1-1)}).$$

Therefore,

$$I_3 = O(\varepsilon + \eta^{\gamma_1} + \eta^{4(\gamma_1-1)})J,$$

where J is defined as

$$J = \int_0^\tau \exp(-2\alpha(\tau - \sigma)) \frac{E}{R_1^3(\sigma)} d\sigma.$$

But J can be more conveniently rewritten as

$$J = \int_0^\tau \exp(-2\alpha(\tau - \sigma)) \ddot{R}_1(\sigma) d\sigma,$$

which we integrate by parts. We find that

$$\begin{aligned} J &= \dot{R}_1(\tau) - e^{-2\alpha\tau} \dot{R}_1(0) + 2\alpha(R_1(\tau) - e^{-2\alpha\tau} R_1(0)) \\ &\quad + 4\alpha^2 \int_0^\tau \exp(-2\alpha(\tau - \sigma)) R_1(\sigma) d\sigma. \end{aligned}$$

We can see now that

$$J = \dot{R}_1(\tau) + O(\eta^{\gamma_1-1}).$$

Therefore

$$\frac{\dot{R}_1(\tau)}{R_1(\tau)} = \left(1 + \frac{O(\eta^{\gamma_1-1})}{R_1(\tau)}\right) (1 + O(\varepsilon + \eta^{\gamma_1} + \eta^{4(\gamma_1-1)})).$$

Since \ddot{R}_1 is nonnegative, for $\tau \in [\eta^3, \tau_1]$, $\dot{R}_1(\tau)$ can be estimated from below by $\dot{R}_1(\eta^3)$ which is equal to

$$\frac{W(\eta^3 - \tau_0)}{\sqrt{(E/W) + W(\eta^3 - \tau_0)^2}};$$

we infer from relations (3.33) and (3.34) that

$$\lim_{k \rightarrow \infty} \dot{R}_1(\eta^3) > 0,$$

which concludes the proof. \square

4. THE CASE $\bar{\theta} < \pi/2$.

We prove here the first theorem which justifies Moreau's rule for $\bar{\theta} < \pi/2$.

Theorem 4.1. *If $\bar{\theta} < \pi/2$, the representative point of the system enters region \mathcal{R}_3 at a time $\bar{t} = t_0 + \bar{\tau}/\sqrt{k}$, where*

$$(4.1) \quad \bar{\tau} \sim \tan \bar{\theta} \sqrt{E/W};$$

moreover, in the coordinates defined by the axes y_1 and y_2 (see Fig. 1), we have the following asymptotics for all $t \geq \bar{t}$:

$$(4.2) \quad 0 \leq y_1(t) \leq \frac{C \exp(\xi_1(t - \bar{t})\sqrt{k})}{\sqrt{k}},$$

$$(4.3) \quad y_2(t) \sim (t - \bar{t})\dot{s}(0) \cos \bar{\theta}.$$

Proof. The differential equation satisfied by Θ defined by (3.28) is deduced from (3.8) and is given by

$$(4.4) \quad \dot{\Theta} = \frac{\sqrt{E}(1 - \varepsilon)}{R^2}, \quad \Theta(0) = 0.$$

Recall that the principal part Θ_1 is defined by (3.29) and (3.30). Let us estimate $\Upsilon_1 = \Theta - \Theta_1$: Υ_1 satisfies the differential equation

$$\dot{\Upsilon}_1 = \frac{\sqrt{E}(1-\varepsilon)}{R^2} - \frac{\sqrt{E}}{R_1^2}.$$

Therefore, if we let $p = \|S_1\|$, and if we denote

$$(4.5) \quad \beta = \frac{\varepsilon + p^2 + 2p}{(1-p)^2},$$

we find that

$$|\dot{\Upsilon}_1| \leq \beta \dot{\Theta}_1.$$

Hence, for all $\tau \in [0, \tau_1]$,

$$(4.6) \quad (1-\beta)\Theta_1(\tau) \leq \Theta(\tau) \leq (1+\beta)\Theta_1(\tau).$$

According to (3.68) and the definition of ε , there exists k_0 such that for all $k \geq k_0$:

$$(4.7) \quad \beta < 1, \quad \bar{\theta} < \frac{\pi(1-\beta)}{2}.$$

Let τ_+ and τ_- be defined by the relations

$$\Theta_1(\tau_+) = \frac{\bar{\theta}}{1-\beta}, \quad \Theta_1(\tau_-) = \frac{\bar{\theta}}{1+\beta}.$$

Thanks to condition (4.7) and formula (3.32) τ_+ and τ_- are well defined, and are given by

$$\begin{aligned} \tau_+ &= \tau_0 + \frac{\sqrt{E}}{W} \frac{\tan(\bar{\theta}/(1-\beta)) - W\tau_0/\sqrt{E}}{1 + (W\tau_0/\sqrt{E}) \tan(\bar{\theta}/(1-\beta))}, \\ \tau_- &= \tau_0 + \frac{\sqrt{E}}{W} \frac{\tan(\bar{\theta}/(1+\beta)) - W\tau_0/\sqrt{E}}{1 + (W\tau_0/\sqrt{E}) \tan(\bar{\theta}/(1+\beta))}. \end{aligned}$$

Therefore, as k tends to infinity, both τ_- and τ_+ are equivalent to $(\sqrt{E}/W) \tan \bar{\theta}$.

The function Θ is strictly increasing with respect to time; thanks to inequality (4.6), there is a unique $\bar{\tau} \in [\tau_-, \tau_+]$ such that $\Theta(\bar{\tau}) = \bar{\theta}$.

We know an equivalent of τ_- and τ_+ as k tends to infinity:

$$\bar{\tau} \sim \frac{\sqrt{E}}{W} \tan \bar{\theta} = O(\eta^2).$$

Together with (3.37), the above relation implies

$$(4.8) \quad R_1(\bar{\tau}) \sim \frac{\dot{r}(0)\eta}{2 \cos \bar{\theta} \sqrt{\alpha^2 - 1}}$$

and from (3.38) that

$$(4.9) \quad \dot{R}_1(\bar{\tau}) \sim \frac{\dot{s}(0) \sin \bar{\theta}}{\eta}.$$

Proposition 3.8 implies $R(\bar{\tau}) \sim R_1(\bar{\tau})$ and $\dot{R}(\bar{\tau}) \sim \dot{R}_1(\bar{\tau})$.

We change coordinates now, taking the axis y_2 along the second side of the convex cone K and the axis y_1 perpendicular to y_2 , and going out of K . The new

time variable is a translation of the natural time, denoted t' , and we set its origin at the time when the representative point enters region \mathcal{R}_3 . We also let $\bar{t} = t_0 + \bar{\tau}/\sqrt{k}$.

With these conventions,

$$y(0) = r(\bar{t}), \quad \dot{y}(0) = \dot{r}(\bar{t}) + ir(\bar{t})\dot{\theta}(\bar{t}).$$

We use now the equivalents obtained previously:

$$(4.10) \quad y_1(0) = O(\eta^2/\sqrt{k}), \quad y_2(0) = 0,$$

$$(4.11) \quad \dot{y}_1(0) \sim \dot{s}(0) \sin \bar{\theta}, \quad \dot{y}_2(0) \sim \dot{s}(0) \cos \bar{\theta}.$$

The second component y_2 of y satisfies the ordinary differential equation

$$\ddot{y}_2 = 0,$$

so that

$$(4.12) \quad y_2(t') \sim t' \dot{s}(0) \cos \bar{\theta}.$$

The first component y_1 of y satisfies the following ordinary differential equation

$$(4.13) \quad \ddot{y}_1 + 2\alpha\sqrt{k}\dot{y}_1 + ky_1 = 0,$$

as long as $y_1 \geq 0$. The explicit solution of (4.13) with initial data (4.10) and (4.11) is given by

$$y_1(t') = \dot{y}_1(0) \frac{e^{\xi_1 t' \sqrt{k}} - e^{\xi_2 t' \sqrt{k}}}{2\sqrt{k}\sqrt{\alpha^2 - 1}} + y_1(0) \frac{\xi_1 e^{\xi_2 t' \sqrt{k}} - \xi_2 e^{\xi_1 t' \sqrt{k}}}{2\sqrt{\alpha^2 - 1}}.$$

Since $\dot{y}_1(0)$ is non negative, $y_1(t')$ stays non negative for all $t' \geq 0$ and we have the following estimate on the first component of y :

$$(4.14) \quad 0 \leq y_1(t') \leq \frac{C e^{-|\xi_1| t' \sqrt{k}}}{\sqrt{k}}.$$

□

Thus, we obtain the conclusion of this section as the following Theorem:

Theorem 4.2. *Let Π_1 be the orthogonal projection on $\{x_1 = 0\}$, and let Π_2 be the orthogonal projection on $\{x_1 \cos \bar{\theta} + x_2 \sin \bar{\theta} = 0\}$; see Fig. 1. As k tends to infinity, u_k converges uniformly on compact sets of \mathbb{R}^+ to u_∞ given by*

$$u_\infty(t) = \begin{cases} u(0) + t\Pi_1 \dot{u}(0) & \text{if } 0 \leq t \leq t_0, \\ (t - t_0)\Pi_2 \Pi_1 \dot{u}(t_0) & \text{if } t_0 \leq t. \end{cases}$$

Proof. The initial part of the motion is described thanks to (2.6) and (2.7). Estimate (4.14) proves that $y_1(t)$ tends to 0 uniformly on compact sets of $]t_0, \infty)$; relation (4.12) enables us to conclude. □

5. THE SECOND ASYMPTOTICS

In the case $\bar{\theta} \geq \pi/2$, we need a new asymptotic, and an estimate which is based essentially on the use of Lyapunov functionals, and which will be proved in Section 6.

We restrict the choice of the exponent γ_1 in the definition of τ_1 by assuming that

$$(5.1) \quad \gamma_1 \in \left(1, \frac{4}{3}\right).$$

The reason for this choice is the following: if (5.1) holds, then the term $E(1 - \varepsilon)^2/R(\tau_1)^3$ is of order $\eta^{3-3\gamma_1}$ which is small relatively to $\dot{R}(\tau_1)$, according to the analysis of 3.8: indeed, the following equivalents of $R(\tau_1)$ and $\dot{R}(\tau_1)$ are a consequence of proposition 3.8:

$$(5.2) \quad R(\tau_1) \sim \dot{s}(0)\eta^{\gamma_1-1}, \quad \dot{R}(\tau_1) \sim \dot{s}(0)\eta^{-1}.$$

Let ζ be such that

$$(5.3) \quad 0 < \zeta < 1/|\xi_1|.$$

We define the time τ_3 by

$$(5.4) \quad \tau_3 = \zeta \ln(1/\eta).$$

We use the notation τ_3 , because we will define below an intermediate time τ_2 between τ_1 and τ_3 .

We claim that the solution of (3.17) on the interval $[\tau_1, \tau_3]$ is very close to the solution of

$$\ddot{R}_2 + 2\alpha\dot{R}_2 + R_2 = 0, \quad R_2(\tau_1) = R(\tau_1), \quad \dot{R}_2(\tau_1) = \dot{R}(\tau_1).$$

Let us define two kernels K_2 and H_2 on \mathbb{R}^+ by

$$K_2(\tau) = \frac{e^{\xi_1\tau} - e^{\xi_2\tau}}{2\sqrt{\Delta}},$$

$$H_2(\tau) = \frac{-\xi_2 e^{\xi_1\tau} + \xi_1 e^{\xi_2\tau}}{2\sqrt{\Delta}}.$$

We extend K_2 and H_2 to \mathbb{R}^- by 0. Therefore, R_2 is given explicitly for $\tau \geq \tau_1$ by

$$(5.5) \quad R_2(\tau) = K_2(\tau - \tau_1)\dot{R}(\tau_1) + H_2(\tau - \tau_1)R(\tau_1).$$

In order to substantiate our claim, we argue as for theorem 3.4: write $R = R_2 + S_2$; then S_2 is a solution of the integral equation

$$(5.6) \quad S_2(\tau) = \int_{\tau_1}^{\tau} K_2(\tau - \sigma) \frac{E(1 - \varepsilon)^2}{(R_2 + S_2)^3(\sigma)} d\sigma.$$

It is convenient to denote

$$\mathcal{L}_2(S_2) = \int_{\tau_1}^{\tau} K_2(\tau - \sigma) \frac{E(1 - \varepsilon)^2}{(R_2 + S_2)^3(\sigma)} d\sigma,$$

whenever $R_2 + S_2$ does not vanish over $[\tau_1, \tau_3]$.

Let us prove that R_2 never vanishes over $[\tau_1, +\infty)$: thanks to the inequalities $0 > \xi_1 > \xi_2$, the functions K_2 and H_2 are positive for $\tau > 0$, $H_2(0)$ is equal to 1; $R(\tau_1)$ and $\dot{R}(\tau_1)$ are strictly positive. Thus the positivity of R_2 is clear.

On the space $C^0([\tau_1, \tau_3])$, we introduce the norm

$$(5.7) \quad \|S_2\| = \sup\{|S_2(\tau)|/R_2(\tau) : \tau \in [\tau_1, \tau_3]\}.$$

We remark that \mathcal{L}_2 is well defined on the open ball of radius 1 about 0 in the norm (5.7).

We prove that \mathcal{L}_2 is a contraction on an appropriate ball, which will lead us to validated asymptotics for R on the interval $[\tau_1, \tau_3]$.

Theorem 5.1. *For all $p \in (0, 1)$, there exists $k_1 > 0$ such that for all $k > k_1$, \mathcal{L}_2 is a contraction from the ball of radius p (relatively to $\|\cdot\|$) about 0 to itself.*

Proof. We will show in Lemma 5.2 that the expression

$$I(\tau) = \frac{1}{R_2(\tau)} \int_{\tau_1}^{\tau} \frac{K_2(\tau - \sigma)}{R_2^3(\sigma)} d\sigma$$

tends to 0 as k tends to infinity, uniformly on $[\tau_1, \tau_3]$.

If $\|S_2\| \leq p < 1$, then

$$|\mathcal{L}_2(S_2)|(\tau) \leq \frac{E(1-\varepsilon)^2}{(1-p)^3} \int_{\tau_1}^{\tau} \frac{K_2(\tau - \sigma)}{R_2^3(\sigma)} d\sigma,$$

and in consequence,

$$\|\mathcal{L}_2(S_2)\| \leq \frac{E(1-\varepsilon)^2}{(1-p)^3} \sup_{\tau_1 \leq \tau \leq \tau_3} I(\tau).$$

Let us estimate $\|\mathcal{L}_2(S_2) - \mathcal{L}_2(\hat{S}_2)\|$ when $\max(\|S_2\|, \|\hat{S}_2\|)$ is at most equal to $p < 1$. We can see that

$$\|\mathcal{L}_2(S_2) - \mathcal{L}_2(\hat{S}_2)\| \leq \frac{3E(1-\varepsilon)^2}{(1-p)^4} \sup_{\tau_1 \leq \tau \leq \tau_3} I(\tau) \|S_2 - \hat{S}_2\|.$$

Therefore, for k large enough, \mathcal{L}_2 is a strict contraction from the ball of radius p about 0 to itself. \square

Let us prove now the estimate announced on I :

Lemma 5.2. *The following estimate holds for $I(\tau)$ on the interval $[\tau_1, \tau_3]$:*

$$(5.8) \quad I(\tau) = O(\eta^{4-3\gamma_1} + \eta^{4(1+\zeta\xi_1)}).$$

Proof. The integral I is analogous to the one defined in (3.48).

We define τ_2 by

$$(5.9) \quad \tau_2 = \frac{2 \ln(\xi_2/\xi_1)}{\xi_1 - \xi_2},$$

and we consider three cases:

- $\tau_1 \leq \tau \leq 2\tau_1$: in this case H_2 cannot be neglected relatively to K_2 .
- $2\tau_1 \leq \tau \leq \tau_2$: in this interval, the dominant term in \dot{R}_2 will be $\dot{R}_2(\tau_1)\dot{K}_2(\tau - \tau_1)$ and an elementary computation shows that this expression vanishes for $\tau = \tau_1 + (\ln(\xi_2/\xi_1))/(\xi_1 - \xi_2)$. Thus \dot{R}_2 crosses 0 approximately at a time $\tau_2/2$.
- $\tau_2 \leq \tau \leq \tau_3$: the last leg of the journey, since K_2 is dominant and in K_2 , the term involving $\exp(\xi_1(\tau - \tau_1))$ is dominant.

Before proving these estimates, we observe that there exist positive numbers M and m such that

$$(5.10) \quad \forall \tau \in \mathbb{R}^+, \quad K_2(\tau) \leq M\tau,$$

$$(5.11) \quad \forall \tau \in [0, \tau_2], \quad K_2(\tau) \geq m\tau.$$

We tackle now the three separate sub-cases in detail.

5.1. First interval: $\tau \in [\tau_1, 2\tau_1]$. We remark that

$$(5.12) \quad R_2(\tau) \geq R(\tau_1)H_2(\tau - \tau_1).$$

Therefore, we can estimate $I(\tau)$ as follows:

$$(5.13) \quad 0 \leq I(\tau) \leq \frac{1}{R_2(\tau)R^3(\tau_1)} \int_{\tau_1}^{\tau} \frac{K_2(\tau - \sigma)}{H_2^3(\sigma - \tau_1)} d\sigma.$$

We observe that over $[\tau_1, 2\tau_1]$,

$$(5.14) \quad H_2(\tau - \tau_1) - 1 = o(\tau_1),$$

and we use (5.10). These observations imply the following inequalities:

$$\begin{aligned} & \frac{1}{R_2(\tau)R^3(\tau_1)} \int_{\tau_1}^{\tau} \frac{K_2(\tau - \sigma)}{H_2^3(\sigma - \tau_1)} d\sigma \\ & \leq C \frac{1}{R^4(\tau_1)} \int_{\tau_1}^{\tau} (\tau - \sigma) d\sigma \\ & \leq C \frac{\tau_1^2}{R^4(\tau_1)} \end{aligned}$$

and thanks to (5.2), the definition (3.46) of τ_1 and condition (5.1), we obtain

$$(5.15) \quad I(\tau) = O(\eta^{4-2\gamma_1}).$$

5.2. Second interval: $\tau \in [2\tau_1, \tau_2]$. We cut the integral I into two pieces: one piece from τ_1 to $2\tau_1$ on which we work essentially as in the previous sub-case, and a piece from $2\tau_1$ to τ_2 on which we work differently. More precisely, on $[\tau_1, 2\tau_1]$, we observe that $R_2(\tau) \geq R(\tau_1)H_2(\tau - \tau_1)$, and on $[2\tau_1, \tau_2]$, $R_2(\tau) \geq \dot{R}(\tau_1)K_2(\tau - \tau_1)$. Therefore,

$$\begin{aligned} & \frac{1}{R_2(\tau)} \int_{\tau_1}^{2\tau_1} \frac{K_2(\tau - \sigma)}{R_2^3(\sigma)} d\sigma \\ & \leq \frac{C}{R_2(\tau)R^3(\tau_1)} \int_{\tau_1}^{2\tau_1} (\tau - \sigma) d\sigma \\ & \leq \frac{C\eta^{3-3\gamma_1}\tau_1\tau}{R_2(\tau)}. \end{aligned}$$

We estimate $K_2(\tau - \tau_1)$ from below by arguing that K_2 increases from 0 to a maximum, and then decreases exponentially fast to 0. Therefore, for all small enough η , there exists τ'_1 tending to infinity such that $K_2(\tau'_1) = K_2(\tau_1)$. Moreover on the interval $[\tau_1, \tau'_1]$ $K_2(\tau)$ is greater than or equal to $K_2(\tau_1)$. Thus, for all large enough k , $K_2(\tau) \geq K_2(\tau_1)$ on the interval $[\tau_1, \tau'_1 - \tau_1]$, and therefore

$$\forall \tau \in [2\tau_1, \tau'_1], \quad R_2(\tau) \geq \dot{R}(\tau_1)K_2(\tau - \tau_1) \geq \dot{R}(\tau_1)K_2(\tau_1).$$

Thus, we obtain thanks to (5.2)

$$(5.16) \quad \frac{1}{R_2(\tau)} \int_{\tau_1}^{2\tau_1} \frac{K_2(\tau - \sigma)}{R_2^3(\sigma)} d\sigma \leq C\eta^{4-3\gamma_1}.$$

For the other piece, we estimate $R_2(\tau)$ from below by $\dot{R}(\tau_1)K_2(\tau - \tau_1)$, and we obtain

$$(5.17) \quad \frac{1}{R_2(\tau)} \int_{2\tau_1}^{\tau} \frac{K_2(\tau - \sigma)}{R_2^3(\sigma)} d\sigma \leq \frac{1}{R_2(\tau)\dot{R}^3(\tau_1)} \int_{2\tau_1}^{\tau} \frac{K_2(\tau - \sigma)}{K_2^3(\sigma - \tau_1)} d\sigma.$$

We use (5.10) and (5.11) in the integral term of the right hand side of inequality (5.17), and we infer that

$$\begin{aligned}
 (5.18) \quad & \frac{1}{R_2(\tau)\dot{R}^3(\tau_1)} \int_{2\tau_1}^{\tau} \frac{K_2(\tau-\sigma)}{K_2^3(\sigma-\tau_1)} d\sigma \\
 & \leq \frac{C}{R_2(\tau)\dot{R}^3(\tau_1)} \int_{2\tau_1}^{\tau} \frac{\tau-\sigma}{(\sigma-\tau_1)^3} d\sigma \\
 & \leq \frac{C(\tau-\tau_1)}{R_2(\tau)\dot{R}^3(\tau_1)\tau_1^2} \\
 & \leq 0(\eta^{4-2\gamma_1}).
 \end{aligned}$$

The combination of (5.16) and (5.18) yields

$$(5.19) \quad I(\tau) = O(\eta^{4-3\gamma_1}).$$

5.3. Third interval: $\tau \in [\tau_2, \tau_3]$. We cut now I into three pieces, relative to the subintervals $[\tau_1, 2\tau_1]$, $[2\tau_1, \tau_2]$ and $[\tau_2, \tau]$.

On the last two pieces, we observe that τ is far from τ_1 , and we use the estimate from below

$$(5.20) \quad R_2(\tau) \geq \dot{R}(\tau_1)K_2(\tau-\tau_1).$$

Moreover there exists C such that for $\tau \geq \tau_2$ and k large enough

$$(5.21) \quad K_2(\tau-\tau_1) \geq C \exp(\xi_1(\tau-\tau_1)).$$

On the first subinterval, i.e. $\sigma \in [\tau_1, 2\tau_1]$, we use inequality (5.10); relations (5.12) and (5.14) imply that

$$R_2^3(\sigma) \geq (1 - o(\tau_1))R(\tau_1)^3;$$

thanks to (5.20) we can see that

$$\frac{1}{R_2(\tau)} \int_{\tau_1}^{2\tau_1} \frac{K_2(\tau-\sigma)}{R_2^3(\sigma)} d\sigma \leq \frac{1 + o(\tau_1)}{R(\tau_1)^3 \dot{R}(\tau_1) K_2(\tau-\tau_1)} \int_{\tau_1}^{2\tau_1} M(\tau-\sigma) d\sigma.$$

Thanks to (5.21) and the asymptotics (5.2), we obtain

$$\frac{1}{R_2(\tau)} \int_{\tau_1}^{2\tau_1} \frac{K_2(\tau-\sigma)}{R_2^3(\sigma)} d\sigma \leq C\eta^{4-3\gamma_1} \exp(-\xi_1(\tau-\tau_1))\tau_1\tau.$$

Since $\tau \leq \tau_3$ and $\exp(-\xi_1\tau_3) = \eta^{\zeta\xi_1}$, we get finally

$$\frac{1}{R_2(\tau)} \int_{\tau_1}^{2\tau_1} \frac{K_2(\tau-\sigma)}{R_2^3(\sigma)} d\sigma \leq C\eta^{4-2\gamma_1+\zeta\xi_1} \ln(1/\eta).$$

Relations (5.3) and (5.1) imply that

$$4 - 2\gamma_1 + \zeta\xi_1 > 0.$$

We observe that

$$(5.22) \quad \forall \sigma \geq 2\tau_1, \quad R_2(\sigma) \geq \dot{R}(\tau_1)K_2(\sigma-\tau_1),$$

and we use estimates (5.10) and (5.11); therefore

$$\int_{2\tau_1}^{\tau_2} \frac{K_1(\tau-\sigma)}{R_2^3(\sigma)} d\sigma \leq \frac{M\tau_3}{\dot{R}^3(\tau_1)} \int_{2\tau_1}^{\tau_2} \frac{d\sigma}{m^3(\sigma-\tau_1)^3} \leq \frac{M\tau_3}{2\dot{R}^3(\tau_1)m^3\tau_1^2}.$$

Now, thanks to (5.20) and (5.21), we obtain

$$\frac{1}{R_2(\tau)} \int_{2\tau_1}^{\tau_2} \frac{K_1(\tau - \sigma)}{R_2^3(\sigma)} d\sigma \leq \frac{C\tau_3}{R(\tau_1)^4 \tau_1^2} \exp(-\xi_1 \tau_3) \leq C\eta^{4-2\gamma_1+\zeta\xi_1} \ln(1/\eta).$$

Let us consider the third piece: we use now estimate (5.21) on the denominator of integrand; since $K_2(\tau) \leq C \exp(\xi_1 \tau)$, and thanks to (5.22), we have

$$\begin{aligned} & \frac{1}{R_2(\tau)} \int_{\tau_2}^{\tau} \frac{K_2(\tau - \sigma)}{R_2^3(\sigma)} d\sigma \\ & \leq \frac{C}{R_2(\tau) \dot{R}^3(\tau_1)} \int_{\tau_2}^{\tau} \frac{\exp(\xi_1(\tau - \sigma))}{\exp(3\xi_1(\sigma - \tau_1))} d\sigma \\ & \leq C\eta^{4(1+\zeta\xi_1)}, \end{aligned}$$

and we conclude that the following estimate holds:

$$(5.23) \quad I(\tau) = O(\eta^{4+\zeta\xi_1-2\gamma_1} \ln(1/\eta) + \eta^{4(1+\zeta\xi_1)}).$$

We have to keep the two terms in the above expression, since we have no way to ascertain the order of the exponents of η .

When we compare the exponents in (5.15) and (5.19), we find that the exponent of η in (5.19) is the smaller; when we look at the exponents in (5.23) to the exponent in (5.19) we find that $4 + \zeta\xi_1 - 2\gamma_1$ is strictly larger than $4 - 3\gamma_1$, and this leads to the conclusion (5.8). \square

We state now the main result of this section:

Proposition 5.3. *The following estimates hold:*

$$(5.24) \quad R(\tau) \sim R_2(\tau) \text{ uniformly over } [\tau_1, \tau_3],$$

$$(5.25) \quad R(\tau_3) \sim \frac{\dot{s}(0)}{2\sqrt{\Delta}} \eta^{-(1+\zeta\xi_1)},$$

$$(5.26) \quad \dot{R}(\tau_3) \sim \frac{\dot{s}(0)}{2\sqrt{\Delta}} \xi_1 \eta^{-(1+\zeta\xi_1)}.$$

Proof. Theorem 5.1 implies the uniform equivalence (5.24), and (5.25) is an immediate consequence of (5.24).

Let us prove an estimate of the derivative \dot{R} at τ_3 :

$$(5.27) \quad \dot{R}(\tau_3) \sim \dot{R}_2(\tau_3) \sim \frac{\dot{s}(0)}{2\sqrt{\Delta}} \xi_1 \eta^{-(1+\zeta\xi_1)}.$$

We observe that

$$\dot{R}(\tau_3) = \dot{R}_2(\tau_3) + \int_{\tau_1}^{\tau_3} \frac{\partial K_2}{\partial \tau}(\tau_3 - \sigma) \frac{E(1-\varepsilon)^2}{(R_2 + S_2)(\sigma)^3} d\sigma.$$

Therefore,

$$|\dot{R}(\tau_3) - \dot{R}_2(\tau_3)| \leq C \int_{\tau_1}^{\tau_3} \left| \frac{\partial K_2}{\partial \tau}(\tau_3 - \sigma) \right| \frac{1}{R_2^3(\sigma)} d\sigma.$$

There exists a constant C such that for all $\sigma \geq 0$

$$\left| \frac{\partial K_2}{\partial \tau}(\sigma) \right| \leq C \exp(\xi_1 \sigma).$$

We use the method which gave estimate (5.23): we cut the integration interval into the three subintervals $[\tau_1, 2\tau_1]$, $[2\tau_1, \tau_2]$ and $[\tau_2, \tau_3]$, and on each of these subintervals, we estimate R_2 from below exactly as in this calculation. Details are left to the reader, and we obtain

$$\int_{\tau_1}^{\tau_3} \frac{\exp(\xi_1 \sigma)}{R_2^3(\sigma)} d\sigma = O(\eta^{3-2\gamma_1-\xi_1\zeta} + \eta^{3(1+\xi_1\zeta)}).$$

The equivalent of $\dot{R}_2(\tau_3)$ is obtained immediately from the explicit formula (5.5) for R_2 and the equivalents (5.2). Hence we infer that (5.27) holds. \square

6. THE FINAL ASYMPTOTICS

In this section, we show that for large enough times $R(\tau)$ is bounded from above. In view of (2.11), this estimate will enable us to show that the angular velocity is bounded from below, and hence, the polar angle Θ will cross through $\bar{\theta}$.

Theorem 6.1. *There exists a strictly positive number R_M and a time τ_4 such that*

$$\forall \tau \geq \tau_4, \quad R(\tau) \leq R_M.$$

Proof. Denote

$$x = \begin{pmatrix} R \\ \dot{R} \end{pmatrix}, \quad N(x) = \begin{pmatrix} 0 \\ E(1-\varepsilon)^2/R^3 \end{pmatrix}, \quad M = \begin{pmatrix} 0 & 1 \\ -1 & -2\alpha \end{pmatrix}.$$

With these notations, equation (3.17) can be written

$$(6.1) \quad \dot{x} = Mx + N(x).$$

We observe that in the domain $(0, \infty) \times \mathbb{R}$, (6.1) has exactly one critical point at

$$x_c = \begin{pmatrix} R_c \\ 0 \end{pmatrix}, \quad R_c = (E(1-\varepsilon)^2)^{1/4}.$$

This critical point is attractive, as an examination of the linearization of (6.1) around x_c shows. Moreover, there is a Lyapunov functional given by

$$(6.2) \quad F(x) = x_1^2 + \frac{E(1-\varepsilon)^2}{x_1^2} + x_2^2.$$

Therefore, given $x(\tau)$ with $x_1(\tau) > 0$, we can see that for all $\tau' \geq \tau$, $F(x(\tau'))$ is at most equal to $F(x(\tau))$, and in particular, $x(\tau')$ remains bounded. We see that when τ tends to infinity, $x(\tau)$ tends to the critical point x_c .

The spectrum of M is $\{\xi_1, \xi_2\}$; therefore, the matrix

$$Q = \int_0^\infty \exp(sM^*) \exp(sM) ds$$

is well defined, symmetric, positive and definite. In particular, if λ_1 is the smallest eigenvalue of Q and λ_2 is the largest eigenvalue of Q ,

$$(6.3) \quad \lambda_1 x^* x \leq x^* Q x \leq \lambda_2 x^* x.$$

If we let $x(t) = e^{tM}x_0$, we observe that

$$\begin{aligned} \frac{d}{dt}x(t)^*Qx(t) &= \frac{d}{dt} \int_0^\infty x_0^* \exp(tM^*) \exp(sM^*) \exp(sM) \exp(tM) x_0 ds \\ &= \frac{d}{dt} \int_t^\infty x_0^* \exp(sM^*) \exp(sM) x_0 ds \\ &= -x(t)^*x(t) \leq -\lambda_2^{-1}x(t)^*Qx(t). \end{aligned}$$

As $x_0 \in \mathbb{R}^2$ and $t \geq 0$ are arbitrary in the above calculation, we have proved indeed that for all $x \in \mathbb{R}^2$

$$(6.4) \quad 2x^*QMx \leq -\lambda_2^{-1}x^*Qx.$$

Since $\dot{x} = Mx + N(x)$ we have the inequality

$$\frac{d}{d\tau}x^*Qx \leq -\frac{x^*Qx}{\lambda_2} + 2(x^*Qx)^{1/2}\sqrt{\lambda_2}|N(x)|.$$

We seek a number \bar{R} such that if $R(\tau) \geq \bar{R}$, then

$$(6.5) \quad \frac{d}{d\tau}(x^*Qx) \leq -\frac{x^*Qx}{2\lambda_2}.$$

Indeed, in order to satisfy (6.5), it suffices to have

$$2\sqrt{\lambda_2}|N(x)|(x^*Qx)^{1/2} \leq \frac{x^*Qx}{2\lambda_2},$$

or equivalently,

$$|N(x)| \leq \frac{(x^*Qx)^{1/2}}{4\lambda_2^{3/2}}.$$

But $|N(x)| = E(1-\varepsilon)^2/R^3$ and $|x| \geq R$, so that, with the help of (6.3), it suffices to satisfy

$$\frac{E}{\bar{R}^3} \leq \frac{\lambda_1^{1/2}\bar{R}}{4\lambda_2^{3/2}},$$

i.e.

$$\bar{R}^4 \geq \frac{4\lambda_2^{3/2}E}{\lambda_1^{1/2}}.$$

We shall show now that if we choose \bar{R} such that

$$\bar{R} > \max \left(\left(\frac{4E\lambda_2^{3/2}}{\lambda_1^{1/2}} \right), R_c \right),$$

then there exists τ_4 such that

$$(6.6) \quad R(\tau_4) = \bar{R}.$$

Indeed, we know from (5.25) that $R(\tau_3) \gg 1$, and that the limit of $R(\tau)$ as τ tends to infinity is R_c ; therefore, $R(\tau)$ must cross \bar{R} . We denote by τ_4 the smallest time in $[\tau_3, \infty)$ such that (6.6) holds.

On the interval $[\tau_3, \tau_4]$, the differential inequality (6.5) implies

$$(x^*Qx)(\tau_4) \leq (x^*Qx)(\tau_3) \exp(-(\tau_4 - \tau_3)/2\lambda_2),$$

whence

$$\frac{\tau_4 - \tau_3}{2\lambda_2} \leq \ln(x^*Qx)(\tau_3) - \ln(x^*Qx)(\tau_4).$$

But $(x^*Qx)(\tau_4) \geq \lambda_1 \bar{R}^2$, and we obtain the inequality

$$\tau_4 \leq \tau_3 + 2\lambda_2 [\ln(x^*Qx)(\tau_3) - \ln(\lambda_1 \bar{R}^2)].$$

In particular, there exists C such that

$$\tau_4 \leq \tau_3 + C \ln(1/\eta).$$

We also need an estimate on $\dot{R}(\tau_4)$. We first show that it is less than or equal to 0. By (5.26) we know that $\dot{R}(\tau_3) < 0$. Denote by (τ_3, τ_5) the connected component of $\{\tau > \tau_3 : \dot{R}(\tau) < 0\}$ whose boundary contains τ_3 .

If $\tau_5 = \infty$, it is clear that $\dot{R}(\tau_4) \leq 0$. Assume that $\tau_5 < \infty$ and that $\dot{R}(\tau_4) > 0$; then $\tau_5 < \tau_4$ and $\dot{R}(\tau_5)$ vanishes.

We infer from differential equation (3.17) that

$$\ddot{R}(\tau_5) = -R(\tau_5) + \frac{E(1-\varepsilon)^2}{R(\tau_5)^3},$$

but $R(\tau_5) > R_c$, because $R(\tau_5) \geq R(\tau_4) = \bar{R}$; therefore

$$(6.7) \quad \ddot{R}(\tau_5) < 0.$$

On the other hand, as $\dot{R}(\tau)$ is negative on (τ_3, τ_5) and vanishes at τ_5 , a straightforward sign argument shows that

$$\ddot{R}(\tau_5) \geq 0,$$

which contradicts (6.7).

Now, we prove that

$$\dot{R}(\tau_4) \geq \xi_1 R(\tau_4).$$

This will be a consequence of the following inequality for all $\tau \geq \tau_1$ and for all large enough k :

$$(6.8) \quad \dot{R}(\tau) - \xi_1 R(\tau) \geq 0.$$

We observe that

$$\frac{d}{d\tau}(\dot{R} - \xi_1 R) = \xi_2(R - \xi_1 R) + \frac{E(1-\varepsilon)^2}{R^3}.$$

When we integrate this differential relation, we find that

$$\begin{aligned} (\dot{R} - \xi_1 R)(\tau) &= \exp(\xi_2(\tau - \tau_1))(\dot{R} - \xi_1 R)(\tau_1) \\ &\quad + \int_{\tau_1}^{\tau} \exp(\xi_2(\tau - \sigma)) \frac{E(1-\varepsilon)^2}{R^3(\sigma)} d\sigma. \end{aligned}$$

For k large enough, the equivalences (5.2) show that $(\dot{R} - \xi_1 R)(\tau_1)$ is strictly positive, and (6.8) follows immediately.

We infer now from (6.8) and the sign condition on $\dot{R}(\tau_4)$ that

$$F(x(\tau_4)) \leq \bar{F} = \bar{R}^2 + E(1-\varepsilon)^2 \bar{R}^{-2} + \xi_1^2 \bar{R}^2.$$

Since the Lyapunov functional decreases along trajectories of the system, we obtain for all $\tau \geq \tau_4$ the inequalities

$$(6.9) \quad \frac{E(1-\varepsilon)}{\bar{F}} \leq R(\tau)^2 \leq \bar{F}, \quad \dot{R}(\tau)^2 \leq \bar{F}.$$

This is the final estimate we needed before the conclusion. \square

We can now state the following corollary relative to the existence of the time $\bar{\tau}$:

Corollary 6.2. *There exists a time $\bar{\tau} \in (0, \infty)$ such that $\Theta(\bar{\tau}) = \bar{\theta}$.*

Proof. We know from (2.11) that Θ is an increasing function of τ ; if there is a time $\bar{\tau} \leq \tau_4$ for which $\Theta(\bar{\tau}) = \bar{\theta}$, the conclusion is clear. Assume otherwise; then, with the notations of (6.9), we can see that

$$\dot{\Theta}(\tau) = \frac{\sqrt{E}(1-\varepsilon)}{R^2} \geq \frac{\sqrt{E}(1-\varepsilon)}{\bar{F}},$$

and the conclusion is also clear. \square

7. THE CASE $\bar{\theta} \geq \pi/2$

In this section we estimate from below the first time $\bar{\tau}$ at which $\Theta(\bar{\tau}) = \bar{\theta}$; we expect that $\bar{\tau}$ will be comparable to τ_1 , but this is not correct. Recall the definition (3.15) of ε ; in this definition, the exponent of η is

$$4\sqrt{\Delta}/|\xi_1|;$$

define a number r by

$$(7.1) \quad r = \min(\gamma_1, 4\sqrt{\Delta}/|\xi_1|).$$

Now, we can state the following theorem:

Theorem 7.1. *If $\bar{\theta} > \pi/2$, then for large enough k , $\bar{\tau} \geq \tau_1$; if $\bar{\theta} = \pi/2$, then there exists a strictly positive number C such that*

$$(7.2) \quad \bar{\tau} \geq C\eta^{\max(2-r, \gamma_1)}.$$

Proof. We argue as follows: assume $\bar{\tau} \leq \tau_1$; we recall estimate (4.6):

$$\forall \tau \in [0, \tau_1], \quad (1-\beta)\Theta_1(\tau) \leq \Theta(\tau) \leq (1+\beta)\Theta_1(\tau),$$

where β is given by (4.5) and $\|S_1\| = p$ satisfies (3.68). The assumption (5.1) implies $4\gamma_1 - 4 < \gamma_1$, and thus (3.68) simplifies as

$$p = O(\eta^{\gamma_1 + \varepsilon}).$$

The definition (7.1) of r implies that

$$\beta = O(\eta^r).$$

Moreover, relation (3.32) leads to

$$\Theta_1(\tau) = \frac{\pi}{2} - \arctan \frac{\sqrt{E}}{W(\tau - \tau_0)} + \arctan \frac{W\tau_0}{\sqrt{E}}.$$

This relation implies immediately that

$$\lim_{k \rightarrow \infty} \Theta_1(\tau_1) = \frac{\pi}{2},$$

and therefore, thanks to (4.6)

$$\lim_{k \rightarrow \infty} \Theta(\tau_1) = \frac{\pi}{2}.$$

If $\bar{\theta} > \pi/2$, the last relation implies immediately that for k large enough, $\bar{\tau}$ is at least equal to τ_1 .

Assume now that $\bar{\theta} = \pi/2$; now, the situation is more delicate, since none of the inequalities established so far implies an estimate on $\bar{\tau}$. If $\bar{\tau} \geq \tau_1$, we are done. Otherwise, we shall estimate $\bar{\tau}$ from below. Already, relation (4.6) implies

$$\Theta_1(\bar{\tau}) \geq \frac{\pi}{2(1 + C\eta^r)},$$

or in other words

$$\frac{\pi}{2} - \arctan \frac{\sqrt{E}}{W(\bar{\tau} - \tau_0)} + \arctan \frac{W\tau_0}{\sqrt{E}} \geq \frac{\pi}{2(1 + C\eta^r)},$$

which implies

$$\arctan \frac{\sqrt{E}}{W(\bar{\tau} - \tau_0)} \leq O(\eta^r),$$

and therefore

$$\bar{\tau} - \tau_0 \geq C\eta^{2-r}.$$

Thus, we have shown that

$$\bar{\tau} \geq C\eta^{2-r}.$$

If $\gamma_1 > 2 - r$, the relations

$$\tau_1 = \eta^{\gamma_1} \geq \bar{\tau} \geq C\eta^{2-r}$$

are contradictory for k large; therefore

$$\gamma_1 > 2 - r, \ k \text{ large} \implies \bar{\tau} \geq \tau_1.$$

Thus, we have shown (7.2). \square

We deduce the following estimates from (7.2) and the asymptotics of sections 3, 5 and 6

$$(7.3) \quad \begin{aligned} C\eta^{\max(1-r, \gamma_1-1)} &\leq R(\bar{\tau}) \leq \frac{C'}{\eta}, \\ |\dot{R}(\bar{\tau})| &\leq \frac{C}{\eta} \end{aligned}$$

We are able to show now the main result of this section:

Theorem 7.2. *For $\bar{\theta} \geq \pi/2$, as k tends to infinity, u_k converges uniformly on the compact sets of \mathbb{R}^+ to u_∞ given by*

$$u_\infty(t) = \begin{cases} u(0) + t\Pi_1 \dot{u}(0) & \text{if } 0 \leq t \leq t_0, \\ 0 & \text{if } t_0 \leq t, \end{cases}$$

where Π_1 is the projection on the line $\{x_1 = 0\}$; see Fig. 1.

Proof. We go back to the original scales and time $\bar{t} = t_0 + \bar{\tau}/\sqrt{k}$; then

$$u_k(\bar{t}) = r(\bar{t})e^{i\bar{\theta}}, \quad \dot{u}_k(\bar{t}) = (\dot{r}(\bar{t}) + ir(\bar{t})\dot{\theta}(\bar{t}))e^{i\bar{\theta}}.$$

Therefore, in coordinates y_1, y_2 (see Fig. 1), we have the relations

$$(7.4) \quad \begin{aligned} y_1(\bar{t}) &= \eta R(\bar{\tau})/\sqrt{k}, \\ y_2(\bar{t}) &= 0, \\ \dot{y}_1(\bar{t}) &= \eta \dot{R}(\bar{\tau}), \\ \dot{y}_2(\bar{t}) &= \eta(1 - \varepsilon)\sqrt{E}/R(\bar{\tau}). \end{aligned}$$

We have also the estimate

$$(7.5) \quad \dot{R}(\bar{\tau}) - \xi_1 R(\bar{\tau}) \geq 0.$$

If $\bar{\tau} \geq \tau_1$, (7.5) is a consequence of (6.8). Otherwise, we observe that $\bar{\tau}$ belongs to $[\eta^3, \tau_1]$ for all large enough k ; therefore, we are able to use the equivalences (3.69) and (3.70), whence

$$\dot{R}(\bar{\tau}) - \xi_1 R(\bar{\tau}) \sim \dot{R}_1(\bar{\tau}) - \xi_1 R_1(\bar{\tau}),$$

which is valid because the dominant term in the right hand side of the above expression does not vanish; indeed, the expression (3.37) of R_1 and (3.38) of \dot{R}_1 , we can see that

$$\dot{R}_1(\bar{\tau}) - \xi_1 R_1(\bar{\tau}) \sim \sqrt{W} - \xi_1 \sqrt{W} \bar{\tau} \sim C\eta^{-1},$$

which implies (7.5); in the original coordinates, (7.5) translates as

$$(7.6) \quad \dot{y}_1(\bar{t}) - \sqrt{k} \xi_1 y_1(\bar{t}) \geq 0.$$

We infer from estimate (7.3) that

$$\begin{aligned} y_1(\bar{t}) &= O(1/\sqrt{k}), \\ \dot{y}_1(\bar{t}) &= O(1), \\ \dot{y}_2(\bar{t}) &= O(\eta^{1-\max(1-r, \gamma_1-1)}) = o(1). \end{aligned}$$

In the coordinates y_1 and y_2 , the system (2.2) can be rewritten

$$(7.7) \quad \ddot{y}_1 + 2\alpha\sqrt{k}\dot{y}_1 + ky_1 = 0$$

as long as $y_1 \geq 0$ and

$$\ddot{y}_2 = 0.$$

But the explicit solution of (7.7) with initial data (7.4) is given by

$$\begin{aligned} y_1(t) &= \dot{y}_1(\bar{t}) \frac{\exp(\xi_1(t - \bar{t})\sqrt{k}) - \exp(\xi_2(t - \bar{t})\sqrt{k})}{2\sqrt{\Delta k}} \\ &\quad + y_1(\bar{t}) \frac{\xi_1 \exp(\xi_2(t - \bar{t})\sqrt{k}) - \xi_2 \exp(\xi_1(t - \bar{t})\sqrt{k})}{2\sqrt{\Delta}}. \end{aligned}$$

If $\dot{y}_1(\bar{t})$ is non negative, it is clear that y_1 stays non negative for all time larger than \bar{t} . If $\dot{y}_1(\bar{t})$ is negative, we use (7.6): we estimate from below $\dot{y}_1(\bar{t})$ by $\sqrt{k} \xi_1 y_1(\bar{t})$, and after simplifications, we get

$$y_1(t) \geq y_1(\bar{t}) \frac{\xi_1 - \xi_2}{2\sqrt{\Delta}} \exp(\xi_1(t - \bar{t})\sqrt{k}).$$

Therefore, (7.7) holds for all $t \geq \bar{t}$. In particular,

$$\forall t \geq \bar{t}, \quad |y_1(t)| = O(1/\sqrt{k}),$$

and

$$\forall t \geq \bar{t}, \quad y_2(t) = O((t - \bar{t})\eta^{1-\max(1-r, \gamma_1-1)}),$$

which proves that in this case the limit of y_1 and y_2 is 0, as k tends to infinity. \square

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